RESEARCH DEPARTMENT

A METHOD FOR SYNTHESIZING THE RADIATION PATTERN OF A RING AERIAL

Report No. E-075

(1962/4)

THE BRITISH BROADCASTING CORPORATION

ENGINEERING DIVISION

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SUMMARY

This report describes a method for determining the currents and voltages with which the radiating elements of a ring aerial must be driven in order to produce a specified radiation pattern, a process often referred to as radiation pattern synthesis. In the method described, the radiation pattern is specified in terms of the field strength required in five horizontal directions. The application of the method to the practical case of a ring of dipoles mounted on a cylinder is illustrated with an example. Although the method does not necessarily lead to the most economical design of aerial it is of interest since it illustrates the difficulties involved in the synthesis of an aerial to meet a prescribed radiation pattern. It is possible that a more economical aerial design may be achieved by modification of the method described in this report; further investigations to this end are contemplated.

1. INTRODUCTION

When designing aerials for very-high-frequency (v.h.f.) broadcasting, care must be taken to ensure that the powers radiated in certain directions do not exceed certain specified values; this is necessary to ensure that interference with other stations broadcasting in the same channel does not occur. At the same time, the aerial must give adequate coverage in its own service area. The maximum permissible and minimum desirable field strengths can be conveniently displayed in polar form as a templet comprising two closed figures. The horizontal radiation pattern (h.r.p.) of a v.h.f. broadcasting aerial must often satisfy somewhat stringent requirements and, although satisfactory radiation patterns can usually be obtained on a trial and error basis by calculating the performance of aerial arrangements which appear to be suitable, this can be a time-consuming process even when an analogue computer is available.[1] The investigation described in this report was undertaken in an attempt to determine the aerial design by direct calculation from the radiation pattern requirements.

The radiation pattern of any aerial, however complex, may be determined if its radiating currents are known, but the converse problem (generally known as synthesis) of finding the current distribution which will produce a given radiation pattern is much more difficult. To simplify the problem, certain assumptions are usually made about the phase variation of the radiation pattern, for example a particular relationship between the feeds to the aerial elements may be postulated. This is undesirable and may lead to an uneconomic aerial design, because the phase variation of a radiation pattern is generally of no practical interest; it is the amplitude variation which is significant.

In the method described in this report no restriction is placed on the phase of the radiation pattern, but the problem is simplified by limiting to five the number of real coefficients which specify the current or voltage distribution. This limitation restricts us to the synthesis of simple radiation patterns; these exist, however, in sufficient variety to enable many practical requirements to be satisfied. More complicated patterns could naturally be obtained by using more than five feed coefficients but the synthesis problem would then be much more difficult to solve.

The method may be applied to rings comprising any type of element whose radiation pattern is symmetrical both in amplitude and phase about a line joining the element with the ring centre. Suitable elements include rings of dipoles placed around a conducting cylinder; the individual dipoles may be either parallel to the cylinder axis or tangential to the circumference of the ring. It is equally applicable to rings of axial or circumferential slots in conducting cylinders.

In applying the method, the magnitude of the required field strength is specified in five directions. Although an infinite number of radiation patterns satisfying the specified values exist, the technique described in this report leads to a single pattern whose shape depends on the relationship between these values. Should the shape of this pattern not be entirely satisfactory for the requirements of the templet in directions other than those with specified field strengths, it may often be improved by making a fresh choice of the five specified values.

When an acceptable pattern has been obtained the current or voltage distribution of the ring aerial is determined. The aerial is initially assumed to contain an infinite number of elements and the required current or voltage distribution derived is therefore a continuous function of the angular position around the ring; it is analogous to the aperture distribution of a linear aerial. In general, four such distributions, all of which result in the same radiation pattern, can be obtained. The continuous distribution is then simulated (in the sense of a 'best approximation') by a finite number of elements with appropriate feeds and the extent to which the pattern deviates from the one originally specified is determined; the amount of deviation which can be tolerated determines the number of elements which are required. It is in this replacement of a continuous distribution of radiating elements by a finite number of elements that the weakness of the method lies. Although it is possible to satisfy a wide variety of templets, the number of radiating elements required is usually greater than the number which are found to give a satisfactory result by the trial and error method.

2. DESCRIPTION OF THE METHOD

Consider a radiating element P on a ring, as shown in Fig. 1. If the radiation pattern of the element in the plane containing the ring is symmetrical in amplitude and phase about OP, the pattern may be referred to the centre of the ring and written in the form

$$K_0 + K_1\cos\phi + K_2\cos 2\phi + \dots + K_m\cos m\phi + \dots \qquad (1)$$

where K_0, K_1, K_2 etc. are complex coefficients and ϕ is the angle measured from the direction OP.

Suppose now that we have a ring of n equally spaced and co-phased radiating elements. Let the complex feed coefficient of each element be u_0/n; this can conveniently correspond to the loop current in the case of a dipole and the loop voltage in the case of a slot. Then if n is sufficiently large, the radiated field will be constant in all directions and equal to u_0K_0. Higher order terms (of order n, $2n$ etc.) may be neglected since the coefficients K_m invariably diminish in amplitude as m increases.

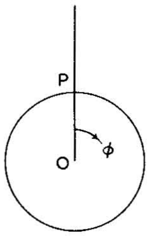

Fig. 1
Co-ordinate system

Consider now a similar ring in which the elements are driven with equal amplitudes but with a phase which advances progressively round the ring by an amount equal to ϕ. If the feed to the element at $\phi = 0$ is u_1/n, the feed to the other elements is $u_1e^{j\phi}/n$ and the radiation pattern is then equal to $\frac{1}{2}u_1K_1e^{j\phi}$, provided sufficient elements are used.*

Similarly we may consider a third ring with an anti-clockwise phase rotation. If the feed* to its elements is $u_{-1}e^{-j\phi}/n$ the radiation pattern will be $\frac{1}{2}u_{-1}K_1e^{-j\phi}$.

If the three rings are now superimposed the feed to the elements is given by

$$u/n = [u_0 + u_1e^{j\phi} + u_{-1}e^{-j\phi}]/n \tag{2}$$

and the radiation pattern will be equal to

$$E = u_0K_0 + \tfrac{1}{2}[u_1e^{j\phi} + u_{-1}e^{-j\phi}]K_1 \tag{3}$$

Since the absolute phase of the radiation pattern is arbitrary we may specify u_0K_0 as a positive real quantity and write the pattern in the form

$$E = A_0 + Qe^{j\phi} + Re^{-j\phi} \tag{4}$$

where A_0 is real and positive and Q and R are complex. Thus

$$A_0 = u_0K_0, \qquad Q = \tfrac{1}{2}u_1K_1, \qquad R = \tfrac{1}{2}u_{-1}K_1 \tag{5}$$

The radiation pattern may also be written in the form

$$E = A_0 + A_1\cos\phi + B_1\sin\phi + j[C_1\cos\phi + D_1\sin\phi] \tag{6}$$

* Although ϕ denotes the azimuthal angle it is also used here to denote the phase of the ring element, because the phase of the element feed is directly related to its angular position.

where A_0, A_1, B_1, C_1 and D_1 are all real. From equations (5) and (6) it may be shown that

$$\begin{aligned} Q &= \tfrac{1}{2}[A_1 + D_1 - j(B_1 - C_1)] \\ R &= \tfrac{1}{2}[A_1 - D_1 + j(B_1 + C_1)] \end{aligned} \tag{7}$$

Since the relative phase of the radiation pattern is of no interest we need consider only the modulus of E or some function of it. The most convenient function is the square of the modulus, which is given by

$$|E|^2 = [A_0 + A_1\cos\phi + B_1\sin\phi]^2 + [C_1\cos\phi + D_1\sin\phi]^2 \tag{8}$$

We may write this expression in the form

$$|E|^2 = a_0 + a_1\cos\phi + a_2\cos 2\phi + b_1\sin\phi + b_2\sin 2\phi \tag{9}$$

where
$$a_0 = A_0^2 + \tfrac{1}{2}(A_1^2 + B_1^2 + C_1^2 + D_1^2) \tag{10}$$

$$a_1 = 2A_0A_1 \quad (11) \qquad a_2 = \tfrac{1}{2}(A_1^2 - B_1^2 + C_1^2 - D_1^2) \tag{12}$$

$$b_1 = 2A_0B_1 \quad (13) \qquad b_2 = A_1B_1 + C_1D_1 \tag{14}$$

Equations (8) and (9) are therefore alternative forms of the power radiation pattern, and since they contain five coefficients, the relative field strength (or the relative power density) may be specified in five different directions.* Now the coefficients in equation (9) can be readily obtained from the specified values by solving five linear simultaneous equations and the radiation pattern may then be calculated for all other values of ϕ from this equation.

It is important to note that any set of coefficients which results in a negative value of $|E|^2$ for any value of ϕ must be rejected, since it does not correspond to a realizable aerial. This is likely to happen if very small field-strength values are specified or if rapid variations of field strength with direction are demanded.** If this occurs, the specified values must be modified until $|E|^2$ is positive for all values of ϕ; the coefficients of equation (9) then satisfy certain conditions which are discussed in the Appendix.

To find the feed coefficients of the elements it is now necessary to determine the coefficients of equation (8) from those of equation (9); the procedure is as follows:

From equations (11), (13) and (14) we have

$$C_1D_1 = b_2 - A_1B_1 = b_2 - \frac{a_1b_1}{4A_0^2}$$

*The absolute field strengths do not need to be specified. All that are required are the ratios between the field strengths in five different directions.

**Although a ring aerial which will satisfy such requirements may be possible, the feed distribution in its elements will not necessarily be of the type described here.

From equations (10) and (12)

$$a_0 + a_2 = A_0^2 + A_1^2 + C_1^2 \tag{16}$$

$$a_0 - a_2 = A_0^2 + B_1^2 + D_1^2 \tag{17}$$

Substituting equation (11) into equation (16) and rearranging, we have

$$C_1^2 = a_0 + a_2 - A_0^2 - \frac{a_1^2}{4A_0^2} \tag{18}$$

Similarly from equations (13) and (17)

$$D_1^2 = a_0 - a_2 - A_0^2 - \frac{b_1^2}{4A_0^2} \tag{19}$$

We may now form and equate the product $C_1^2 D_1^2$ from equations (15), (18) and (19), giving

$$\left[b_2 - \frac{a_1 b_1}{4A_0^2} \right]^2 = \left[a_0 + a_2 - A_0^2 - \frac{a_1^2}{4A_0^2} \right] \left[a_0 - a_2 - A_0^2 - \frac{b_1^2}{4A_0^2} \right] \tag{20}$$

The term $[a_1 b_1/4A_0^2]^2$ is common to both sides of this equation and cancels, leaving a cubic in A_0^2. In the Appendix it is shown that a practical solution is possible only if this equation has three real roots, in which case they are all positive, but only the two smaller correspond to physically realizable solutions to the problem. The equation therefore yields two values for A_0, which by definition is real and positive.

The remaining coefficients may now be determined from equations (11), (13), (18) and (19). For each value of A_0 there are two values (positive and negative) of C_1 and D_1 but as the signs of C_1 and D_1 are related to each other by equation (14), there are only two solutions for each value of A_0, or four in all. Thus the radiation pattern of equation (9) can be achieved by four different feed distributions; their calculation is straightforward, Q and R being determined from equations (7) and the feed coefficients from equations (5).

3. APPLICATION OF THE METHOD TO RINGS OF DIPOLES MOUNTED ON A CYLINDER

An application which is of particular interest is the synthesis of the h.r.p. of a v.h.f. broadcasting aerial. These aerials often consist of rings of dipoles placed round a mast; in many practical cases the latter may be assumed to be equivalent to an infinitely long conducting cylinder.

A templet typical of those prescribed for Band I television transmitting aerials is shown in Fig. 2; the radiation pattern of the aerial is required to lie between the two closed curves.

Investigation showed that by a suitable choice of the five specified field-strength values it was possible to obtain a pattern which satisfied the templet in all directions; the values which were chosen are given in Table 1 below.

TABLE 1

Specified values

ϕ	0	45°	135°	180°	270°
$\lvert E \rvert$	1·00	1·00	0·30	0·30	0·50
$\lvert E \rvert^2$	1·00	1·00	0·09	0·09	0·25

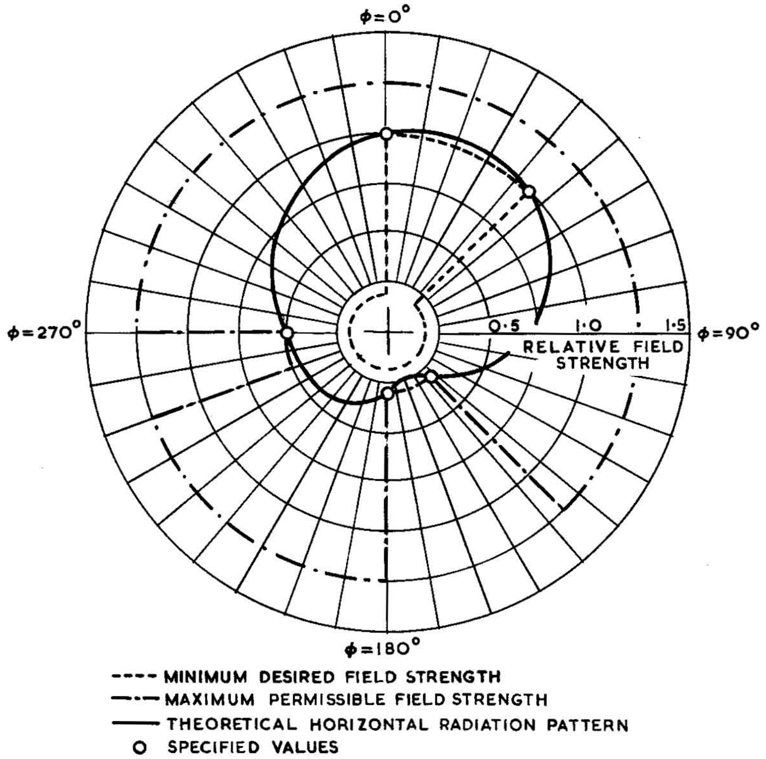

Fig. 2 - Radiation pattern satisfying prescribed templet

By substituting these values in equation (9) and solving the resulting five simultaneous equations, the following values for the coefficients were obtained:

$$a_0 = 0{\cdot}458$$

$$a_1 = 0{\cdot}455 \quad b_1 = 0{\cdot}122$$

$$a_2 = 0{\cdot}087 \quad b_2 = 0{\cdot}134$$

These coefficients satisfy the conditions for which a solution can be found (as stated in the Appendix) and give the radiation pattern shown in Fig. 2.

Substitution of these values in equation (20) gives a cubic in A_0^2 with three real roots. In the Appendix it is shown that the two smaller roots lead to a physically realizable solution of the problem; their values are 0·1234 and 0·2733 and the corresponding values of A_0 are 0·3513 and 0·5228.

Each value of A_0 leads to a pair of solutions; the four solutions are designated 1(a), 1(b), 2(a) and 2(b) and the corresponding coefficients are given in Table 2.

TABLE 2

Radiation Pattern Coefficients

	Solution Number			
	1(a)	1(b)	2(a)	2(b)
A_0	0·3513		0·5228	
A_1	0·6476		0·4352	
B_1	0·1736		0·1167	
C_1	0·0467	−0·0467	0·2870	−0·2870
D_1	0·4663	−0·4663	0·2899	−0·2899

It is of interest at this stage to calculate the radiation patterns, in complex form, from equation (6). When plotted on the complex plane they appear as ellipses, as shown in Fig. 3. It will be seen that the amplitudes of all four radiation patterns are identical and correspond to Fig. 2. The two patterns corresponding to a particular value of A_0 differ only in that their imaginary values are of opposite sign, but the two values of A_0 give quite different ellipses; it will be noted that the origin is inside the ellipse in one case and outside it in the other.

The coefficients of Q and R of equation (4) are now calculated from equations (7); their values are given in Table 3.

TABLE 3

Solution No.	Q	R
1(a)	$0{\cdot}5570 - j0{\cdot}0635$	$0{\cdot}0906 + j0{\cdot}1101$
1(b)	$0{\cdot}0906 - j0{\cdot}1101$	$0{\cdot}5570 + j0{\cdot}0635$
2(a)	$0{\cdot}3626 + j0{\cdot}0852$	$0{\cdot}0726 + j0{\cdot}2018$
2(b)	$0{\cdot}0726 + j0{\cdot}2018$	$0{\cdot}3626 - j0{\cdot}0852$

The coefficients given in Tables 2 and 3 define, in alternative ways, a radiation pattern passing through the five prescribed points.

Up to this point no mention has been made of the type of radiating element, since the aperture distribution can be obtained from any type of radiating element whose radiation pattern satisfies the requirements of symmetry. The element pattern must now be taken into consideration to enable the feed distribution to be determined.

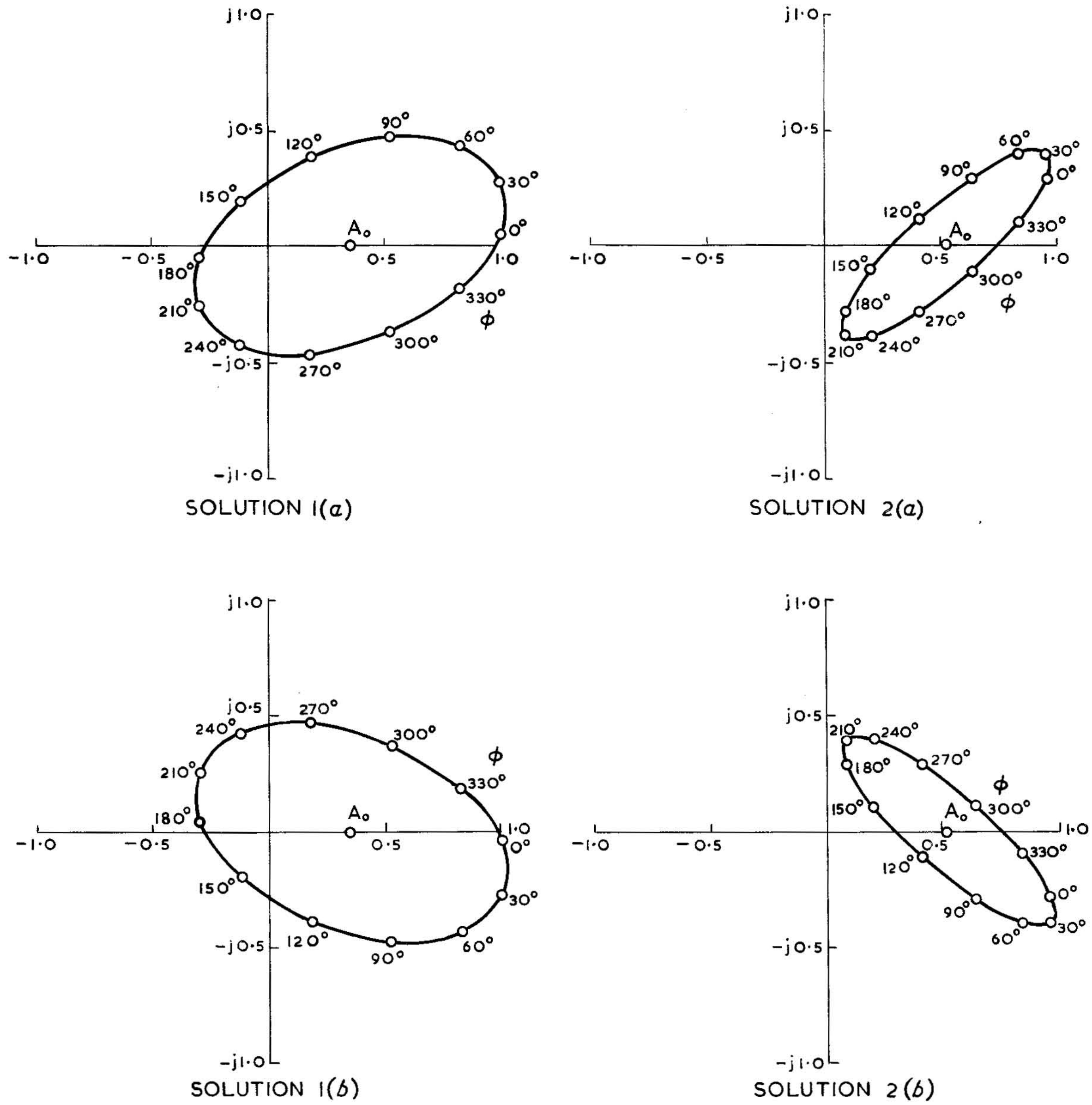

Fig. 3 - Horizontal radiation patterns in complex form

Let us suppose that the aerial consists of a ring of vertical dipoles mounted on a mast, the latter being assumed to be equivalent to a conducting cylinder. The radiation pattern of a single element has been shown[2] to be

$$V_0 + 2 \sum_{n=1}^{\infty} j^n V_n \cos n\phi$$

where

$$V_n = J_n(\beta b) - J_n(\beta a) \frac{H_n^{(2)}(\beta b)}{H_n^{(2)}(\beta a)}$$

$H_n^{(2)}$ denotes the Hankel function of the second kind of order n, b is the distance of the dipole from the cylinder axis, a is the radius of the cylinder and β is the propagation constant of free-space. Comparing this expression with equation (1) it is seen that the complex coefficients of the element radiation pattern are given by

$$K_0 = V_0 \qquad K_1 = j2V_1$$

If $\beta a = 0{\cdot}5$ and $\beta b = 2{\cdot}0$ radians (typical values for a Band I aerial), K_0 and K_1 have the following values:

$$K_0 = 0{\cdot}2385 + j0{\cdot}0535 \qquad K_1 = -0{\cdot}1792 + j1{\cdot}0886$$

The feed coefficients for the three superimposed ring aerials may now be calculated from equations (5); they have the values given in the table below.

TABLE 4

Feed coefficients for vertically polarized aerial

Solution No.	u_0	u_1	u_{-1}
1(a)	$0{\cdot}270 - j0{\cdot}570$	$-0{\cdot}278 - j0{\cdot}978$	$0{\cdot}170 - j0{\cdot}195$
1(b)	$0{\cdot}270 - j0{\cdot}570$	$-0{\cdot}223 - j0{\cdot}129$	$-0{\cdot}050 - j1{\cdot}015$
2(a)	$0{\cdot}402 - j0{\cdot}848$	$0{\cdot}046 - j0{\cdot}674$	$0{\cdot}339 - j0{\cdot}189$
2(b)	$0{\cdot}402 - j0{\cdot}848$	$-0{\cdot}382 - j0{\cdot}070$	$-0{\cdot}259 - j0{\cdot}624$

These four sets of coefficients enable the four feed distributions to be calculated using equation (2). Each distribution is the sum of the currents or voltages with which the three basic ring aerials must be driven in order to give the radiation pattern shown in Fig. 2; they may therefore be regarded as cylindrical aperture distributions. Thus a ring aerial consisting of an infinite number of elements driven according to one of these distributions will have a radiation pattern which is identical with that shown in Fig. 2.

The four aperture distributions are shown in complex form in Fig. 4 and it will be seen that these again take the form of two pairs of ellipses. Since a finite number of elements must be used in practice, curves of the type shown in Fig. 4 enable the most convenient arrangement of elements to be chosen. For example, Fig. 4 shows that in solution 2(a) the current required in an element at $\phi = 180^\circ$ is almost zero and in a practical aerial this element could be omitted altogether. This solution therefore offers a definite advantage over the other three solutions.

The current in each element may be either the average value for the sector of the aperture which contains the element, or it may be the value for the actual position of the element; the latter value is the more convenient since it may be immediately determined from Fig. 4.

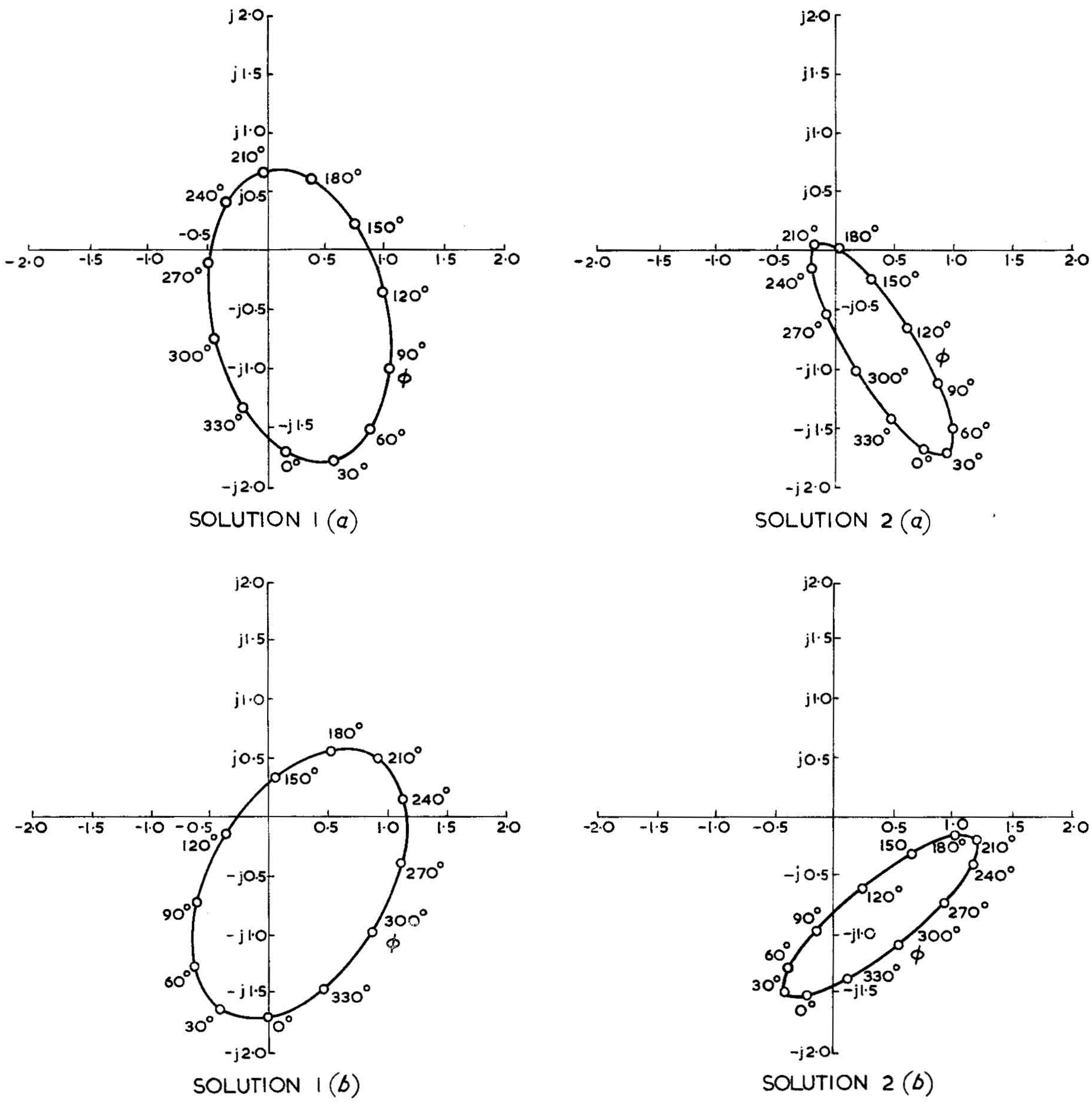

Fig. 4 - Aperture distributions

The use of a finite number of elements means that the resulting pattern will deviate from that shown in Fig. 2. To illustrate the accuracy with which the idealized pattern may be reproduced, a number of patterns for different numbers of elements fed with currents corresponding to their actual positions (rather than values averaged over a sector) were calculated on an analogue computer;[1] the results are shown in Fig. 5. It will be seen that the agreement obtained with the use of eight elements is good, with six it is reasonable but with four it is rather poor. The number of elements which must be used to obtain good agreement depends on the ring radius and with a larger ring more elements will be required.

The necessity for using an almost continuous ring of elements to obtain a close approximation to the required radiation pattern can possibly be overcome by

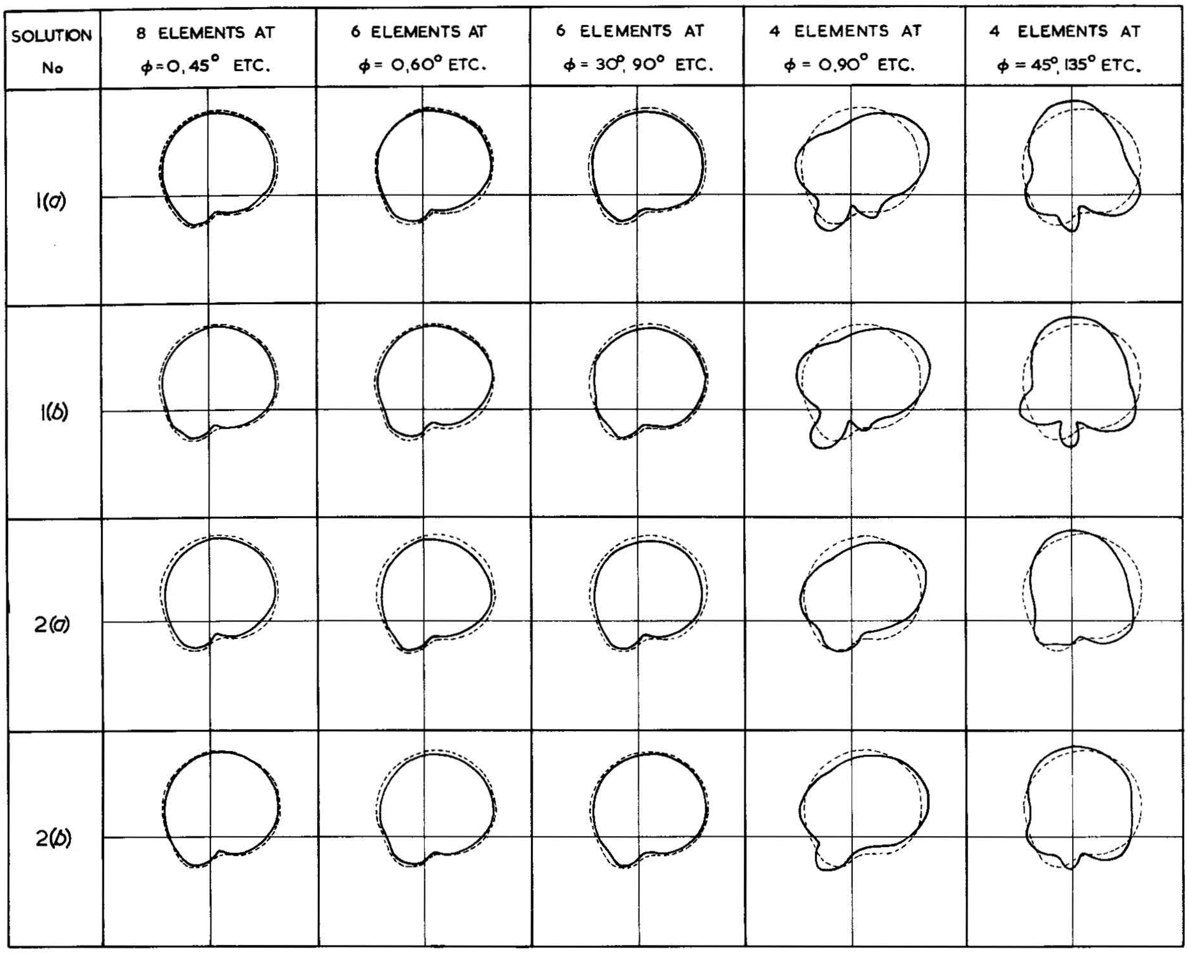

Fig. 5 - Horizontal radiation patterns obtained with finite numbers of elements

The broken line shows the radiation pattern which would be obtained with an infinite number of elements

using an alternative approach. In this method a finite number of elements would be postulated and their feed currents or voltages denoted by complex coefficients. The expression for the resulting radiation pattern (with higher-order terms neglected) would then be compared with each of the four solutions of equation (6) in turn, thereby yielding four sets of feed coefficients. Whether this method is feasible and if so, whether the approximations involved cause the resulting pattern to differ appreciably from the specified pattern, has yet to be investigated.

4. CONCLUSIONS

The design of television transmitting aerials is usually carried out on a trial and error basis; h.r.p. calculations are repeated, for different radiating element arrangements, until a satisfactory result is obtained.

An alternative synthesis method is described in this report which enables the feed parameters for ring aerials to be determined when the field strength is specified in five arbitrarily chosen directions. The method may be used for v.h.f. broadcasting aerials consisting of rings of dipoles or slots, when directional h.r.p.s are required. A single radiation pattern results from the method but this may be obtained with any of four different sets of feed parameters; the most convenient of these feed arrangements may thus be chosen.

The fact that the synthesis method leads to several solutions for the feed parameters suggests that a number of solutions can also be found by the trial and error method. Only one solution is usually found in this way, however, and it may not be the most convenient one to achieve in practice; the choice of solutions offered by the synthesis method is therefore an advantage. The fact that the method leads to a continuous aperture distribution is a disadvantage, since it requires that each ring should contain a sufficient number of elements to approximate to a continuous ring. Moreover it does not reveal the fact that the desired radiation pattern can sometimes be achieved by a much simpler aerial system. For this reason its usefulness, when applied to the design of Band I and II aerials, i.e. those comprising rings of less than one wavelength diameter, is somewhat limited, and the trial and error method is preferable. In the u.h.f. band, however, where larger rings may be required, the method may have a useful application.

5. REFERENCES

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2. Carter, P.S., "Antenna Arrays around Cylinders", Proc. I.R.E., Vol.31, No.12, p.671, December 1943.

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APPENDIX

The conditions which the radiation pattern coefficients must satisfy in order that a solution may be found

A solution to the synthesis problem discussed in this report can only be found if $|E|^2$ is positive for all values of ϕ. The coefficients of equation (9) must therefore satisfy certain conditions which are discussed below.

Equation (9) may be written in the form

$$|E|^2 = a_0 + k_1\cos\theta_1 + k_2\cos\theta_2 \tag{21}$$

where

$$k_1 = \sqrt{a_1^2 + b_1^2} \qquad k_2 = \sqrt{a_2^2 + b_2^2}$$

and θ_1 and θ_2 are functions of ϕ and of the coefficients of equation (9); $\cos\theta_1$ and $\cos\theta_2$ can have any value between −1 and 1.

The value of $|E|^2$ cannot be less than $a_0 - k_1 - k_2$ and will therefore always be positive if $a_0 > k_1 + k_2$. On the other hand, it can be easily shown that $|E|^2$ is bound to be negative for some values of ϕ if the difference between k_1 and k_2 is greater than a_0. However, if a_0 lies in the range

$$|k_1 - k_2| < a_0 < k_1 + k_2$$

it is possible for $|E|^2$ to be positive for all values of ϕ; this case will therefore be considered in greater detail.

First it is necessary to determine which of the roots of equation (20) correspond to valid solutions. This equation is a cubic in A_0^2 and it may have either one or three real roots. Negative roots must be excluded since A_0 is by definition real.

The right-hand side of the equation is the product of C_1^2 and D_1^2; these quantities are given by equations (18) and (19). The behaviour of C_1^2 as a function of A_0^2 is shown in Fig. 6; the behaviour of D_1^2 is similar. Since C_1 and A_0 are real by definition, a solution is only possible if C_1^2 is positive when A_0^2 is positive. From equation (18) it may be shown that this will occur if a_1^2 is less than $(a_0 + a_2)^2$; C_1^2 is then positive when A_0^2 lies in the range

$$\tfrac{1}{2}\left[a_0 + a_2 - \sqrt{(a_0 + a_2)^2 - a_1^2}\right] < A_0^2 < \tfrac{1}{2}\left[a_0 + a_2 + \sqrt{(a_0 + a_2)^2 - a_1^2}\right]$$

Similarly D_1^2 must be positive for some positive values of A_0^2; this will occur when A_0^2 lies in the range

$$\tfrac{1}{2}\left[a_0 - a_2 - \sqrt{(a_0 - a_2)^2 - b_1^2}\right] < A_0^2 < \tfrac{1}{2}\left[a_0 - a_2 + \sqrt{(a_0 - a_2)^2 - b_1^2}\right]$$

provided b_1^2 is less than $(a_0 - a_2)^2$.

A valid solution, however, can occur only if C_1^2 and D_1^2 are simultaneously positive when A_0^2 is positive. In other words, the two ranges for A_0^2 stated above must

overlap. It may be shown that this condition is satisfied if

$$4a_2^2 < \left[\sqrt{(a_0 + a_2)^2 - a_1^2} + \sqrt{(a_0 - a_2)^2 - b_1^2}\right]^2$$

The product of C_1^2 and D_1^2, which is the right-hand side of equation (20), is plotted in Fig. 7 as a function of A_0^2, together with the left-hand side of equation (20). The intersections of the two curves correspond to the roots of this equation; the cases in which there are one and three real roots are illustrated separately.

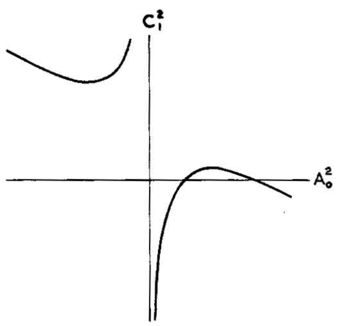
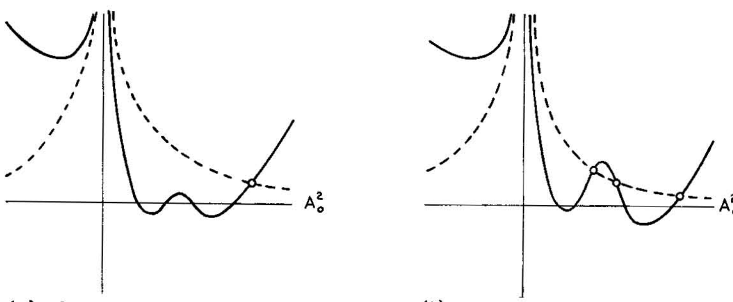


Fig. 6

Fig. 7 shows that the larger of the three real roots, or the single real root when there is only one, occurs when C_1^2 and D_1^2 are both negative* and must therefore be excluded. Thus valid solutions can only occur when equation (20) has three real roots. This is not a sufficient condition because three real roots are still possible even when the positive ranges of C_1^2 and D_1^2 do not overlap. However, if the positive ranges of C_1^2 and D_1^2 do overlap, a solution will exist provided equation (20) has three positive real roots.



(a) One real root (b) Three real roots

Fig. 7

——— r.h.s. of equation (20)
– – – – l.h.s. of equation (20)

Equation (20) may be written in the form

$$A_0^6 + \lambda A_0^4 + \mu A_0^2 + \nu = 0 \tag{22}$$

where

$$\lambda = -2a_0$$

$$\mu = a_0^2 - a_2^2 - b_2^2 + \tfrac{1}{4}(a_1^2 + b_1^2)$$

$$\nu = -\tfrac{1}{4}\left[a_1^2(a_0 - a_2) + b_1^2(a_0 + a_2) - 2a_1b_1b_2\right]$$

This is a cubic in A_0^2 which may have either three real roots or one real and two complex roots.

*Their product is therefore positive.

Equation (22) may be transformed into an equation of the form

$$x^3 + qx + r = 0 \tag{23}$$

where
$$q = \mu - \frac{\lambda^2}{3}$$
$$r = \frac{2\lambda^3}{27} - \frac{\lambda\mu}{3} + \nu$$

by making the substitution $A_0^2 = x - \lambda/3$. Now λ is real, since $\lambda = -2a_0$ and a_0 is real; thus if x is real the corresponding value of A_0^2 must also be real. Consequently, if equation (23) has three real roots, equation (22) must also have three real roots.

Equation (23) is the standard form of the cubic equation and it may be shown[3] that it has three real roots if

$$\left(\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3 < 0 \tag{24}$$

It may also be shown that, if this condition is satisfied, the three roots must lie in the range

$$-\sqrt{-q/3} \leqslant x \leqslant \sqrt{-q/3} \tag{25}$$

It should be noted that, if condition (24) is satisfied, q must be negative. Consequently $\sqrt{-q/3}$ is real; the positive root is assumed.

Since $A_0^2 = x - \lambda/3$, it follows that the three roots of equation (22) must lie within the range

$$-\sqrt{-q/3} - \lambda/3 \leqslant A_0^2 \leqslant \sqrt{-q/3} - \lambda/3 \tag{26}$$

Thus the roots will all be positive if

$$-\sqrt{-q/3} - \lambda/3 > 0 \tag{27}$$

Substituting for q we may write this inequality in the form

$$-\frac{\lambda}{3}\left[\sqrt{1 - \frac{3\mu}{\lambda^2}} + 1\right] > 0 \tag{28}$$

Since $\sqrt{-q/3}$ is real and positive, $\sqrt{1-3\mu/\lambda^2}$ must also be real and positive. Thus the term inside the bracket in (28) is positive. Now $\lambda = -2a_0$, and a_0 is positive, since the case in which a_0 is greater then $|k_1 - k_2|$ is being considered.* λ is therefore negative, both the factors of (28) are positive and the inequality is satisfied. Equation (22) therefore has three positive real roots, of which the two smaller correspond to valid solutions to the problem.

The conditions which determine whether the problem has a solution may be summarized as follows:

* k_1 and k_2 are defined below equation (21).

If $a_0 > k_1 + k_2$ a solution always exists

If $a_0 < |k_1 - k_2|$ a solution is not possible

If $|k_1 - k_2| < a_0 < k_1 + k_2$ a solution can be found if the following conditions are satisfied

$$a_1^2 < (a_0 + a_2)^2$$

$$b_1^2 < (a_0 - a_2)^2$$

$$4a_2^2 < \left[\sqrt{(a_0 + a_2)^2 - a_1^2} + \sqrt{(a_0 - a_2)^2 - b_1^2} \right]^2$$

$$\left(\frac{r}{2}\right)^2 + \left(\frac{q}{3}\right)^3 < 0$$

where q and r are functions of a_0, a_1, etc. and may be deduced from the relations given given below equations (22) and (23).

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